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RADIAL POINT INTERPOLATION COLLOCATION METHOD (RPICM) FOR THE SOLUTION OF TWO PHASE FLOW THROUGH POROUS MEDIA

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ABSTRACT

In this paper, a kind of meshfree (meshless) methods, namely Radial Point Interpolation Collocation Method (RPICM), has been applied to the numerical simulation of two phase flow in porous media. The main feature of this approach is to use the interpolation schemes in local supported domains based on radial basis functions. As a result, this present method is different from the traditional global Radial basis functions (RBF) method because it is local and its algebraic matrix is banded. In the numerical testing, it was applied to solve a 2D homogeneous reservoir problem. Some preliminary numerical simulation results, which will be beneficial for us to further investigate reservoir simulations by the present method, have been obtained. Meanwhile, the feature with no meshes needed shows the present schemes possess a considerable perfect potential to reservoir simulation and environmental engineering problems.

NOMENCLATURE

$\mathbf{v}_w$, $\mathbf{v}_o$ (m.s^{-1}) the velocities of water and oil phases
p_w, p_o (Pa) the pressures of water and oil
P_c (Pa) capillary pressure
μ_w, μ_o (kg.m^{-1}.s^{-1}) the dynamic viscosities
ρ_w, ρ_o (kg.m^{-3}) the densities of water and oil phases
S_w water saturation
λ_o, λ_w the phase mobilities for water and oil phases
ϕ the porosity of medium
$\tilde{q}_w, \tilde{q}_o$ the sources of water and oil phases
k_{ro}, k_{rw} the relative permeabilities
$\mathbf{v_T}$ is the total velocity

INTRODUCTION

There are many approaches applied to reservoir numerical simulation in petroleum industry. To name a few: Finite Difference Method (FDM); Finite Element Method (FEM), etc. (Peaceman, 1977; Aziz, 1979) and in recent years, "Meshfree (Meshless) Methods" have been attracting a great attention in the field of computational mechanics and computational mathematics. A variety of approaches named as "meshfree (meshless)" have appeared (Liu G. R., 2002). Here, a kind of meshfree (meshless) methods, namely Radial Point Interpolation Collocation Method (RPICM), has been applied to the numerical simulation of two phase immiscible flow in porous media. Point Interpolation Method (PIM) was originally proposed by Liu and Gu (1999), PIM is based on Galerkin or Petrov-Galerkin weak forms, and numerical integrations are required. Radial PIM (RPIM) was presented by Liu and Wang (2002). RPICM has also been developed by Liu and Tang (2003).

In this paper, RPICM is applied to 2D reservoir simulation. In next section, the governing equations for immiscible flow are briefly derived, and then point interpolation approximations with Radial basis functions are constructed. Meanwhile, collocation schemes for the pressure and saturation equations are formulated with RPICM. Then, this present numerical formulations are employed to solve immiscible flow in reservoir simulation. Finally some concluding remarks are given.

GOVERNING EQUATION FOR TWO-PHASE IMMISCIBLE FLOW

The governing equations describing two-phase immiscible flow through porous media are considered below. Two immiscible fluids involve two separate phases: the wetting phase (water) and the non-wetting phase (oil).
Water-phase:

$$-\nabla\cdot(\rho_w\mathbf{v_w}) = \phi\frac{\partial}{\partial t}(\rho_w S_w) + \tilde{q}_w \tag{1}$$

Oil-phase:

$$-\nabla\cdot(\rho_o\mathbf{v_o}) = \phi\frac{\partial}{\partial t}\left[\rho_o(1-S_w)\right] + \tilde{q}_o \tag{2}$$

S_w is the water saturation; ρ_w, ρ_o are the density of the water and the oil, respectively; ϕ is the porosity of medium; $\tilde{q}_w, \tilde{q}_o$ .are the sources of water and oil phases. $\mathbf{v_w}, \mathbf{v_o}$ are the velocities of water and oil phases.
If ignoring the effect of gravity, the phase velocities are usually expressed through Darcy's Law as follows:

$$\mathbf{v_w} = -\lambda_w\nabla p_w,\ \mathbf{v}_o = -\lambda_o\nabla p_o \tag{3}$$

λ_o, λ_w are the phase mobilities for water and oil phases.

$$\lambda = \lambda_o + \lambda_w, \lambda_o = \frac{kk_{ro}}{\mu_o}, \lambda_w = \frac{kk_{rw}}{\mu_w} \tag{4}$$

where k is the absolute permeability, k_{ro}, k_{rw} are the relative permeabilities, μ_o, μ_w are the dynamic viscosities, p_w, p_o are the pressures of water and oil and they can be expressed as:

$$p_o = p, p_w = p - P_c \tag{5}$$

Where P_c is the capillary pressure, and assumed here to be a function of saturation.

Assuming that the fluids and the medium are incompressible, continuity (or "pressure") equation can be yielded:

$$\nabla \cdot \mathbf{v_T} = \nabla \cdot (\mathbf{v_w} + \mathbf{v_o}) = -(q_o + q_w) = -q_T \tag{6}$$

$$\mathbf{v_T} = \mathbf{v_w} + \mathbf{v_o}, q_o = \tilde{q}_o / \rho_o, q_w = \tilde{q}_w / \rho_w \tag{7}$$

Where $\mathbf{v_T}$ is the total velocity. It can be derived by equations (3), (5) and (7) as follows:

$$\mathbf{v_T} = -(\lambda_w + \lambda_o)\nabla p - \lambda_w \left(-\frac{dP_c}{dS_w} \right) \nabla S_w \tag{8}$$

$$\mathbf{v_o} = f_o \mathbf{v_T}, \quad \mathbf{v_w} = f_w \mathbf{v_T} \tag{9}$$

We defined:

$$f_o = \frac{\lambda_o}{\lambda_w + \lambda_o}, f_w = \frac{\lambda_w}{\lambda_w + \lambda_o}, \bar{\lambda} = \frac{\lambda_w \lambda_o}{\lambda_w + \lambda_o} \tag{10}$$

$$\nabla P_c = \frac{dP_c}{dS_w} \nabla S_w \tag{11}$$

$$\nabla \cdot (f_o v_T) = \mathbf{v_T} \cdot \nabla f_o + f_o \nabla \cdot \mathbf{v_T}$$
$$= \mathbf{v_T} \cdot \frac{df_o}{dS_w} \nabla S_w - f_o q_T \tag{12}$$

$$f_w + f_o = 1 \rightarrow \frac{df_o}{dS_w} = -\frac{df_w}{dS_w} \text{ and}$$

$$-q_o + f_o q_T = q_w - f_w q_T$$

Substituting equations (8), (9), (10) and (11) into (1), saturation equation can be obtained:

$$\phi \frac{\partial S_w}{\partial t} + \mathbf{v_T} \frac{df_w}{dS_w} \cdot \nabla S_w + \nabla \cdot \left[\bar{\lambda} \frac{dP_c}{dS_w} \nabla S_w \right] \tag{13}$$
$$= f_w q_T - q_w$$

If $P_c = 0$, we obtained Buckley-Leverett equation for saturation:

$$\phi \frac{\partial S_w}{\partial t} + \mathbf{v_T} \frac{df_w}{dS_w} \cdot \nabla S_w = f_w q_T - q_w \tag{14}$$

If $P_c = 0$, $p_w = p_o = p$. The phase velocities can be expressed as:

$$\mathbf{v_w} = -\lambda_w \nabla p, \ \mathbf{v}_o = -\lambda_o \nabla p \tag{15}$$

In this case, the pressure equation can be derived from (6) and (15):

$$-\nabla \cdot (\lambda \nabla p) = q_T \tag{16}$$

At sink well, a commonly used condition is

$$q_w = f_w q_T, q_o = f_o q_T \tag{16a}$$

However, at source well,

$$q_T = q_w \rightarrow q_w - f_w q_T = (1 - f_w) q_w \neq 0 \tag{16b}$$

RADIAL POINT INTERPOLATION

The approximation of a field function $u(x)$, using radial basis functions, may be written as a linear combination of n radial basis functions, viz.,

$$u(\mathbf{x}) \cong \hat{u}(\mathbf{x}) = \sum_{i=1}^{n} a_i \phi(\|\mathbf{r} - \mathbf{r}_i\|, c_i) \tag{17}$$

where n is the number of points in the supported domain near $\boldsymbol{x}$, a_i are coefficients to be determined and ϕ are the radial basis function, such as Multiquadrics (MQ), Gaussian, Thin plate spline.

MQ function: $\phi(\|\mathbf{r} - \mathbf{r}_i\|, c_i) = \left(\sqrt{\|\mathbf{r} - \mathbf{r}_i\|^2 + {c_i}^2} \right)^Q$;

The shape parameter can be defined as $c_i = c = (\alpha_c r_c)$ (Liu G. R., 2002).

For 2-D problems:

$$\|\mathbf{r} - \mathbf{r}_i\| = \sqrt{(x - x_i)^2 + (y - y_i)^2} \tag{18}$$

Due to good accuracy and convergence of MQ, MQ is chosen as interpolation basis function in following numerical calculations. When MQ function is adopted, the selection of the shape parameter c is an issue. In a given interpolation, as c value increases, the shape of the MQ interpolants becomes flatter, and the collocation matrix becomes more singular. The accuracy of the approximation, however, gets better and better, until the numerical inversion breaks down due to round off error. How to choose the optimal shape parameter is a problem that has received the attention of many researchers (Liu and Wang, 2002). So far, this is an open question and no mathematical theory has been developed for determining the optimal value. Here, the form of dimensionless shape parameter α_c will be employed. The r_c is the characteristic length that is related to the nodal space in the local supported domain of the collocation point and it is usually the average nodal spacing for all the nodes in this supported domain. Here, for uniform discrete model, it is chosen to be the distance between uniformly equally distributed points. For scattered point model, it is chosen to be average size of supported domain.

The coefficients a_i in Equations (17) can be determined by enforcing that the function interpolations pass through all n nodes within the supported domain.

The interpolations of the function at the kth point have the form:

$$\hat{u}(\mathbf{x}_k) = a_1 \phi(\|\mathbf{r}_k - \mathbf{r}_1\|, c) + \cdots + a_n \phi(\|\mathbf{r}_k - \mathbf{r}_n\|, c),$$
$$k = 1, \cdots n \tag{19}$$

They can be expressed by matrix formulations as follows:

$$\hat{\mathbf{u}}^e = \mathbf{\Phi a} \tag{20}$$

$$\hat{\mathbf{u}}^e = \left[\hat{u}(\mathbf{x}_1) \quad \cdots \quad \hat{u}(\mathbf{x}_k) \quad \cdots \quad \hat{u}(\mathbf{x}_n) \right]^T \tag{21}$$

$$\mathbf{a} = \left[a_1 \quad \cdots \quad a_i \quad \cdots \quad a_n \right]^T \tag{22}$$

Thus the unknown coefficients vector

$$\mathbf{a} = \mathbf{\Phi}^{-1} \hat{\mathbf{u}}^e \tag{23}$$

Finally, the approximation form of function can be obtained as follows:

$$\hat{u}(\mathbf{x}) = \boldsymbol{\varphi a} = \boldsymbol{\varphi \Phi}^{-1} \hat{\mathbf{u}}^e = \boldsymbol{\psi} \hat{\mathbf{u}}^e \tag{24}$$

$$\boldsymbol{\varphi} = \left[\phi(\|\mathbf{r} - \mathbf{r}_1\|, c) \quad \cdots \quad \cdots \quad \phi(\|\mathbf{r} - \mathbf{r}_n\|, c) \right] \tag{25}$$

The shape functions can be expressed as follows

$$\boldsymbol{\psi} = \boldsymbol{\varphi}\boldsymbol{\Phi}^{-1} = \begin{bmatrix}\psi_1 & \cdots & \psi_i & \cdots & \psi_n\end{bmatrix} \tag{26}$$

Here ψ_i are shape functions. The shape functions constructed have delta function property, which is very attractive to impose essential boundary condition in Galerkin-based meshfree methods (Liu and Gu, 2001; Liu and Wang 2002).

RPICM FOR PRESSURE AND SATURATION EQUATIONS

From (24), the point interpolation forms for pressure and saturation functions can be expressed as:

$$\hat{p}(\mathbf{x}) = \boldsymbol{\psi}\hat{\mathbf{p}}^e,\ \frac{\partial \hat{p}(\mathbf{x})}{\partial \mathbf{x}} = \frac{\partial \boldsymbol{\psi}}{\partial \mathbf{x}}\hat{\mathbf{p}}^e,\ \frac{\partial^2 \hat{p}(\mathbf{x})}{\partial \mathbf{x}^2} = \frac{\partial^2 \boldsymbol{\psi}}{\partial \mathbf{x}^2}\hat{\mathbf{p}}^e \tag{27}$$

$$\hat{S}_w(\mathbf{x}) = \boldsymbol{\psi}\hat{\mathbf{S}}_w^e,\ \frac{\partial \hat{S}_w(\mathbf{x})}{\partial \mathbf{x}} = \frac{\partial \boldsymbol{\psi}}{\partial \mathbf{x}}\hat{\mathbf{S}}_w^e,\ \frac{\partial^2 \hat{S}_w(\mathbf{x})}{\partial \mathbf{x}^2} = \frac{\partial^2 \boldsymbol{\psi}}{\partial \mathbf{x}^2}\hat{\mathbf{S}}_w^e \tag{28}$$

Assuming that there are N_d internal (domain) points and $N_b = N_{b1} + N_{b2}$ boundary points, where N_{b1} are Neumann boundary points and N_{b2} are Dirichlet boundary points. At time t^{m+1}, the following N_d equations are satisfied in internal domain nodes:

$$-\nabla\cdot(\lambda^{m+1}\nabla\hat{p}_i^{m+1}) = 0 \tag{29}$$

$$\phi\frac{(\hat{S}_w^{m+1})_i - (\hat{S}_w^m)_i}{\Delta t} + \mathbf{v}_T^{m+1}\frac{df_w^{m+1}}{d\hat{S}_w^{m+1}}\cdot\nabla\hat{S}_w^{m+1} = \varepsilon\Delta S_w^{m+1}, i = 1,2,\cdots N_d \tag{30}$$

It should be noted that a small diffusion term has been added to (30) to handle shocks and ε is the adjustable diffusion coefficient. Here, implicit time integration scheme has been adopted.

The following $2*N_{b1}$ equations are satisfied on reflected boundaries for both pressure and saturation equations:

$$\mathbf{n}^T\cdot\nabla\hat{p}_i^{m+1} = 0\ ,\ i = 1,\cdots\cdots N_{b1} \tag{31}$$

$$\mathbf{n}^T\cdot\nabla\hat{S}_{iw}^{m+1} = 0\ ,\ i = 1,\cdots\cdots N_{b1} \tag{32}$$

The following $2*N_{b2}$ equations are satisfied on Dirichlet boundaries for both pressure and saturation:

At source well for injected fluid (water),

$$\hat{p}_i^{m+1} = p_{i0}\ ,\ \hat{S}_{iw}^{m+1} = 1.0\ ,\ i = 1,\cdots\cdots N_{b2} \tag{33}$$

At sink well for produced fluid (oil),

$$\hat{p}_i^{m+1} = 0\ ,\ \hat{S}_{iw}^{m+1} = 0.0\ ,\ i = 1,\cdots\cdots N_{b2} \tag{34}$$

$\hat{p}_i, \hat{S}_{iw}$, and their derivatives in equations (29-34) can be obtained by (27) and (28) while $\mathbf{x}=\mathbf{x}_i$. The last system algebraic equation can be solved by Newton-Raphson iteration scheme.

NUMERICAL SIMULATIONS

In this section, a 2D homogeneous two-phase immiscible reservoir problem (Langtangen, 1990) has been tested by RPICM. Owing to symmetry, only the upper half of the reservoir needs to be discretized. Source and sink nodes with specified p values are indicated by the thick lines in Figure 1. S_w was kept equal to unity at the source. Values of the physical parameters are as in Figure 1. The 25×13=325 uniform and 325-point Halton scattered point models shown as Figure 2 and Figure 3 are used to solve the pressure and saturation equations. Figure 4 shows the wetting fluid saturation distributions and its contours at 1000 days and 2000 days with uniform model. In this case, the following MQ parameters were chosen, that is $c=\alpha_c c_0$, $\alpha_c=3.0$, $c_0=16.0$m, $Q=1.0$. There are 25 points in every supported domain. Figure 5 shows the wetting fluid saturation distributions and its contours at 1000 days and 2000 days with random model. In this case, the following MQ parameters were chosen, that is $c=\alpha_c c_0$, $\alpha_c=0.5$, c_0=59.16m is the average size of all supported domains, $Q=1.0$. There are 30 points in every supported domain. For both uniform model and random model, the following time integration parameters have been chosen: 500 time step numbers; 4 days time interval. In addition, diffusion coefficient ε=0.01 was chosen for two cases. The results obtained by RPICM in this paper are compared with Fig. 13 in Langtangen's paper (1990), and they approximately tend to be the same. It should be noticed that results obtained with uniform and scattered point models show some difference at water saturation front. The reason perhaps is due to the choice of shape parameter and supported domain. Further research and explanation will be completed in future.

CONCLUSIONS

We have employed a novel numerical method RPICM to solve immiscible two-phase flow in porous media. Some preliminary numerical simulation results, which will be beneficial for us to further investigate reservoir simulations by the present method, have been obtained. The reason that the results shown in Figure 4 and Figure 5 are some different perhaps is due to the choice of shape parameter and supported domain. In a word, further perfect numerical simulations will be our future research topics, especially for the simulation when using scattered point model. In addition, the issue regarding the *h*-convergence depending on the shape parameter in MQ function need to be further investigated for RPICM.

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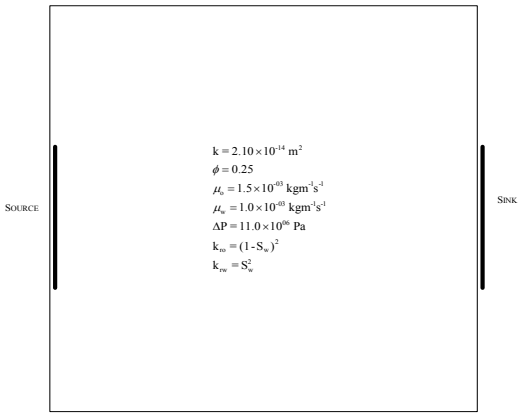


Figure 1: 2D horizontal homogeneous reservoir.
The size of the reservoir is [0,400.0m]× [0,400.0m]

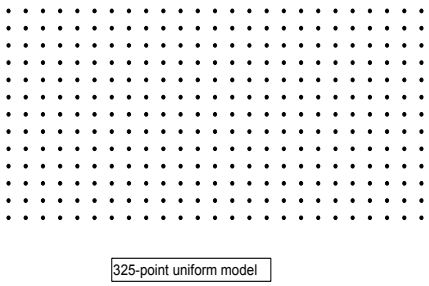


Figure 2: 25×13=325 uniform model

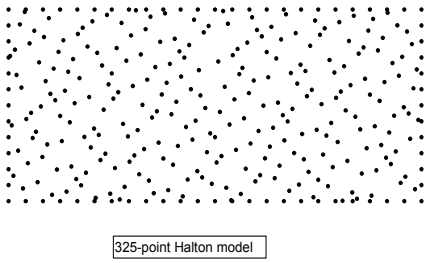
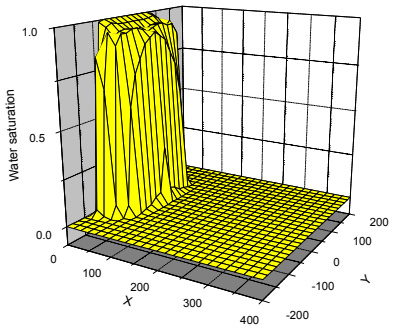
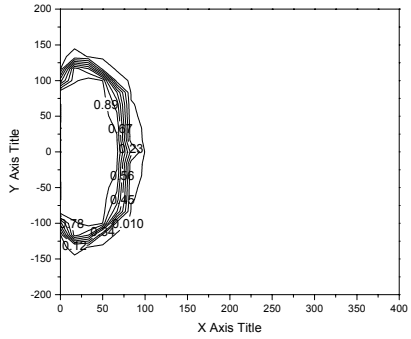
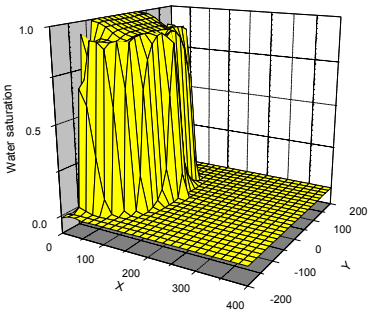


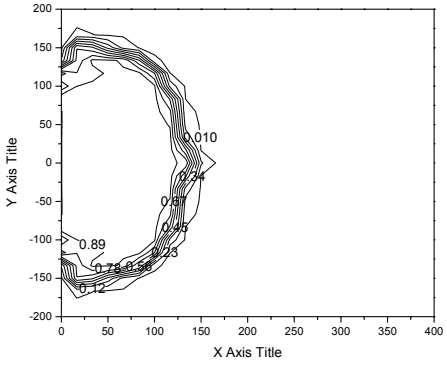
Figure 3: 325-point Halton scattered model





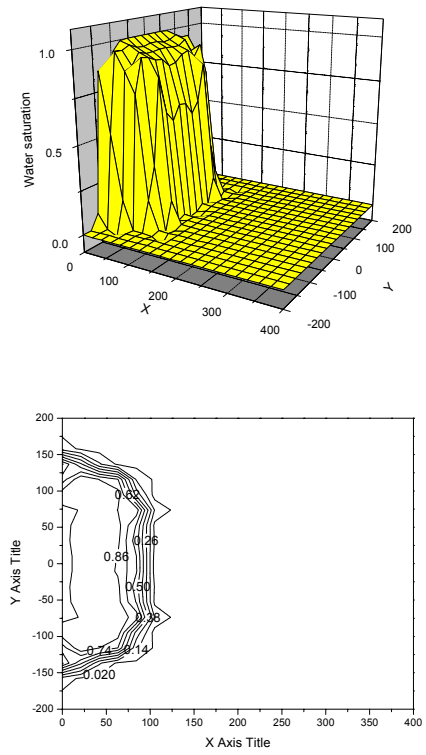
(a) at 1000 days



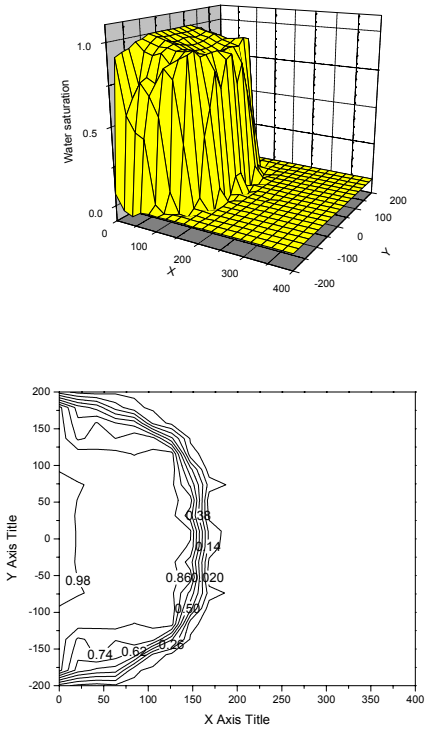


(b) at 2000 days

Figure 4: Water saturation distribution for 25×13=325 uniform model



(a) at 1000 days



(b) at 2000 days

Figure 5: Water saturation distribution for 325-point Halton random model