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STABILITY OF OSCILLATORY AND PULSATILE PIPE FLOW

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ABSTRACT

Linear flow stability analysis is concerned with the long time-limit behaviour of a fluid-dynamical system, namely the system's tendency to find a steady state, or make a transition to a turbulent one. It is known that steady laminar flow in pipes (Hagen–Poiseuille flow) is linearly stable to general perturbations at finite Reynolds numbers, and that single-harmonic oscillatory flow is stable to axisymmetric perturbations. We extend the analysis to demonstrate that the pure oscillatory flow is also stable to general perturbations. We explain how this implies that all laminar steady and periodic circular pipe flows of this type are linearly stable. The least stable modes identified in this study are axially invariant.

NOMENCLATURE

K_n axial pressure gradient at frequency harmonic n
D pipe diameter
J_0 Bessel function of order 0.
k (integer) azimuthal wavenumber
L_z axial domain length
n (integer) frequency harmonic
p pressure
r radial position in pipe
$\Re$ Real component
R pipe radius
Re Reynolds number
t time
T pulse period
$\boldsymbol{u}$ velocity vector
u axial velocity component
$\overline{u}$ area-average flow speed
U_{red} reduced velocity
Wo Womersley number
α axial wavenumber
ρ fluid density
ν fluid kinematic viscosity

INTRODUCTION

Oscillatory (zero-mean) and pulsatile (non-zero mean) incompressible flows in a straight, rigid, circular tube are canonical phenomena of classical fluid mechanics. In addition they serve as models of a variety of flows of engineering and physiological application, for example peristaltic pumping and arterial flows.

As is well known, steady laminar flow in a circular tube (Hagen–Poiseuille flow) is linearly stable to general infinitesimal disturbances for all Reynolds numbers yet studied (e.g. Drazin and Reid, 1981; Schmid and Henningson 1994) but is observed to become turbulent at bulk flow Reynolds numbers of order 2000–3000 in moderately careful experiments. There is still debate about the precise mechanism that leads to transition.

For single-harmonic oscillatory pipe flow, linear stability analysis for axisymmetric perturbations by Yang and Yih (1977) also suggests linear stability to perturbations of this kind for all Reynolds numbers and periods of oscillation. As for steady pipe flow however, experiments (e.g. Eckmann and Grotberg 1991) show that transition to turbulence can occur, often in the form of bursts during each oscillation.

In the case of pulsatile pipe flow (when an oscillation is superimposed on a steady mean flow), experiments, e.g. by Stettler and Fazle Hussain (1986), also demonstrate the presence of burst-type transition. However in this case, there is presently no published study of linear stability, a deficiency that our present work aims to remedy. As we explain below, it is however sufficient to study the problem of linear stability of oscillatory flows, since the stability of the steady flow component is well established.

Potentially the pulsatile pipe-flow transition region (as parameterised either by a Womersley or Reynolds number) can be used to define the operational limits of peristaltic devices in biological applications where low shear stresses are advantageous. Here, high throughput is required, although turbulent transition presents a problem as suspensions, colloids and blood may be damaged. Transition to turbulence also is of interest in arterial flows as turbulence can induce high spatio-temporal gradients of wall shear stress, leading to changes in morphology of endothelial cells and disturbing normal regulation of transport of blood-borne chemicals to and from the arterial walls.

BASE FLOWS AND PROBLEM PARAMETERS

All axisymmetric laminar steady and time-periodic laminar flows of Newtonian fluids in straight circular pipes driven by an axially constant pressure gradient can be decomposed into a sum of Bessel–Fourier solutions. We examine the linear stability of axisymmetric and axially uniform oscillatory and pulsatile flow of fluid with density ρ and kinematic viscosity ν in a circular pipe of diameter D. These base flows are obtained in closed form as analytical Bessel–Fourier solutions first published by Sexl (1930) and later by Womersley (1955):

$$u_n(r,t) = \Re\left[\frac{K_n iT}{\rho 2\pi n}\left(\frac{J_0(i^{3/2}Wo2\,r/D)}{J_0(i^{3/2}Wo)} - 1\right)\exp 2\pi i n\, t/T\right], \quad (1)$$

where $Wo = (2\pi/T\nu)^{1/2}D/2$ is a dimensionless frequency parameter known as the Womersley number, n is a frequency harmonic, J_0 is a complex Bessel function and K_n is an associated complex axial pressure gradient amplitude. In the limit as T grows without bound, this analytical solution asymptotes to the standard parabolic Hagen–Poiseuille solution for steady laminar flow in a circular pipe.

An important point to be made about the base flows under consideration is that any temporally periodic axisymmetric laminar pipe flow (including steady flow, the infinite-period case) can be expressed as a linear sum of terms of type (1). Since all the flows have the same boundary conditions, we can consider their linear stability on a term-by-term basis, one term for each temporal Fourier harmonic. Again exploiting linearity, we can deal with general spatial perturbations at each temporal period as a linear sum of axial and azimuthal Fourier modes, with wavenumbers $\alpha = 2\pi D/L_x$ and k respectively. One implication is that we do not here need to examine the linear stability of the steady flow, since that has been comprehensively dealt with in previous works – it suffices to examine the stability of the oscillatory components, and these can be dealt with one temporal harmonic at a time.

The area-average or bulk flow speed is a function of time:

$$\bar{u}(t) = \left(8/D^2\right)\int_0^{D/2} u(r,t) r dr \cdot \qquad (2)$$

Without loss of generality we can ignore the pressure gradient K_n as a parameter and adjust the phases and amplitudes of the solutions (1) such that at each temporal harmonic n, we have

$$\bar{u}_n(t) = A_n \cos(2\pi nt/T) + B_n \sin(2\pi nt/T) \cdot \qquad (3)$$

The n=0 case corresponds to the standard Hagen–Poiseuille solution $u(r) = A_0 2[1-(r/R)^2]$, and as stated above, is also a solution to (1).

In the present application it is useful to define the peak bulk flow speed as

$$\bar{u}_p = \max_{0<t\le T} \bar{u}(t) \cdot$$

Dimensionless parameters

The flows in question generally have two dimensionless parameters that describe the pulse period and some measure of the flow speed. Taking $\bar{u}_p$ as a velocity scale and diameter D as a length scale, the time scale is $D/\bar{u}_p$. This leads to one choice of the two dimensionless parameters as a Reynolds number and reduced velocity, respectively

$$\mathrm{Re} = \frac{\bar{u}_p D}{\nu} \quad \text{and} \quad U_{red} = \frac{\bar{u}_p T}{D} \cdot$$

In the steady flow case, the only parameter is the Reynolds number, *Re.* Alternatively, for the oscillatory components, the viscosity can be confined to the Womersley number

$$Wo = \sqrt{\frac{\pi D^2}{2\nu T}},$$

and the mean flow velocity scale can again appear through the reduced velocity. This pairing is a sensible choice for the oscillatory cases in that the Womersley number appears in the analytical solution for the base flows. We note that the oscillatory components of the base flow have (via eq. 1) radial velocity profiles that are only a function of *Wo*, *r/D,* and *t.* The reduced velocity is then a premultiplying kinematic factor that describes how far the bulk flow oscillates along the pipe, expressed in pipe diameters, but does not alter the velocity profile. For oscillatory flows we nonetheless would expect *a priori* that their stability could be a function of the two dimensionless flow parameters, *Wo* and U_{red}, as well as axial and azimuthal wavenumbers.

In Figure 1 we show radial profiles of axial velocity at ten phase-points in the base flow cycle and the bulk flow speed as a function of time for two Womersley numbers. With increasing *Wo*, the velocity profile becomes more like plug flow but with small overshoots near the pipe wall. This behaviour was first noted by Richardson in 1929 and studied analytically by Sexl (1930).

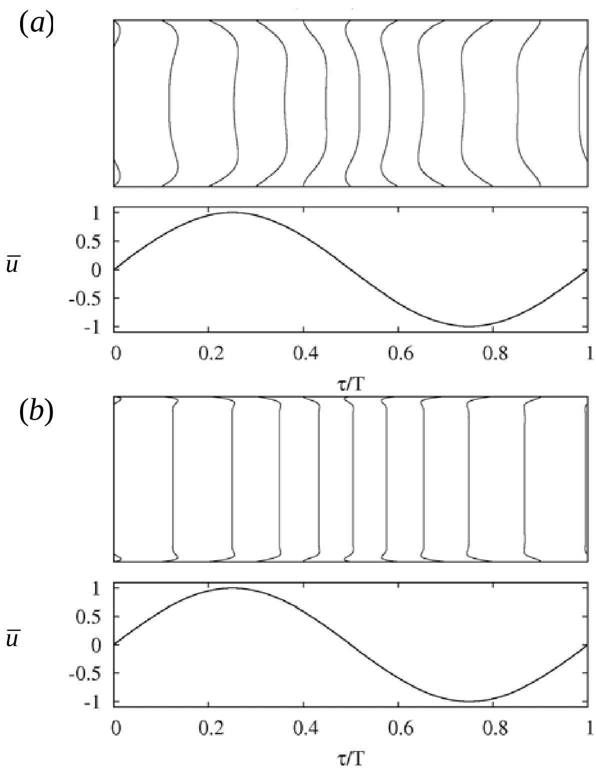


Figure 1: Shows radial velocity profiles at different base-flow phases – note the reversed flow lags in the near-wall region by up to 90°. The area-average flow speed $\bar{u}(t)$ is shown as a function of time for oscillatory base flows with (a) *Wo*=10.23; (b) *Wo*=35.5.

NUMERICAL METHODS

Stability analysis

The stability analysis problem is solved in primitive variables. Starting from the incompressible Navier–Stokes equations

$$\partial_t u = -u \bullet \nabla u - \nabla p + \nu \nabla^2 u, \quad \nabla \bullet u = 0\,,$$

where p is the kinematic or modified pressure, it is proposed that $\boldsymbol{u}$=$\boldsymbol{U}$+$\boldsymbol{u}'$ where $\boldsymbol{U}$ is the base flow whose stability is examined and $\boldsymbol{u}'$ is an infinitesimal perturbation. Upon substitution and retaining terms linear in $\boldsymbol{u}'$, the linearized Navier–Stokes equations are obtained:

$$\partial_t u' = -u' \bullet \nabla U - U \bullet \nabla u' - \nabla p' + \nu \nabla^2 u' \qquad (4)$$

We note that in the present problem, the base flow is T-periodic, i.e. $\boldsymbol{U}(t+T)=\boldsymbol{U}(t)$. Because in incompressible flows the pressure is not an independent variable, and all terms are linear in $\boldsymbol{u}'$, we can write this evolution equation in symbolic form

$$\partial_t u' = L(u')$$

where L is a linear operator with T-periodic coefficients through the influence of the base flow. Correspondingly the stability analysis of this equation is a linear temporal Floquet problem (Iooss & Joseph 1990). Writing the state evolution of $\boldsymbol{u}'$ over one period as

$$u'(t+T) = A(T)u'(t) \qquad (5)$$

where $A(T)$ is the system monomodry matrix, we obtain a Floquet eigenproblem

$$A(T)u''_j(t) = \mu_j u''_j(t),$$

where $u''_j(t)$ are phase-specific Floquet modes and μ_j are Floquet multipliers (which in general occur in complex-conjugate pairs). Stability of the problem is assessed from the Floquet multipliers: unstable modes have multipliers that lie outside the unit circle in the complex plane (i.e. $|\mu| > 1$), while stable modes lie inside (i.e. $|\mu| < 1$).

We use a time-stepping based methodology outlined in Tuckerman and Barkley (2000), given detailed explanation in Barkley, Blackburn and Sherwin (2008), and previously used in studies of various oscillatory flows (e.g. Blackburn 2002; Blackburn *et al.* 2005; Blackburn and Sherwin 2007) in order to solve the Floquet eigenproblem. A key point about the approach is that a system monomodry matrix $A(T)$ is not explicitly constructed; rather, a Krylov method is used that is based on repeated application of the state transition operator (5) whose action is obtained by integrating the linearised Navier–Stokes equations forward in time over interval T. By varying the Krylov dimension and ensuring sufficient resolution we are typically able to resolve a moderate number (e.g. four) of the leading (least stable) Floquet modes. Here we have in the main concentrated on the dominant mode. All the multipliers for the leading modes found here are real, and positive.

As noted above, time-periodic base flows enter the problem through the linearised advection terms in equation (4) – the base flows are precomputed at a moderate number of phase points or time-slices and then may be accurately reconstructed during timestepping via Fourier interpolation in time. Since the base flows in the present problem are analytically defined, this reconstruction could be avoided, however it is fast, cheap and accurate when enough time-slices are used – in the present work we have found 64 time-slices are typically sufficient for accuracy. A small number of leading Floquet multipliers and eigenmodes are obtained via Krylov subspace projection. This is followed by Ritz reconstruction of modes on the full solution space. The reader is directed to the references supplied above for further detail.

Discretization

Spatial discretization and time integration is handled using a cylindrical coordinate spectral element method with mixed explicit/implicit time stepping, as outlined in Blackburn and Sherwin (2004). The domain is discretized into spectral elements in the meridional semi-plane that runs from the pipe axis to the outer radius in the radial direction and a finite length of pipe L_z in the axial direction, as shown for example in Figure 2**.**

Fourier modal structure is assumed in the azimuthal direction with integer wavenumbers k, and as a result of linearization, each azimuthal mode can be dealt with independently. In the axial direction we use real wavenumbers $\alpha = 2\pi D/L_z$. Because of the approach taken to spatial discretization in the axial direction, the Floquet eigensolution for any domain length can contain modes for both $\alpha = 0$ (i.e. modes that are axially invariant) and $\alpha = m2\pi D/L_z$ (where m is an integer). Typically, there are a number of multipliers for α=0 that are larger in magnitude than the first axially variant mode and we compute sufficient modes to be assured that we obtain the leading mode for $\alpha=2\pi D/L_z$ as well.

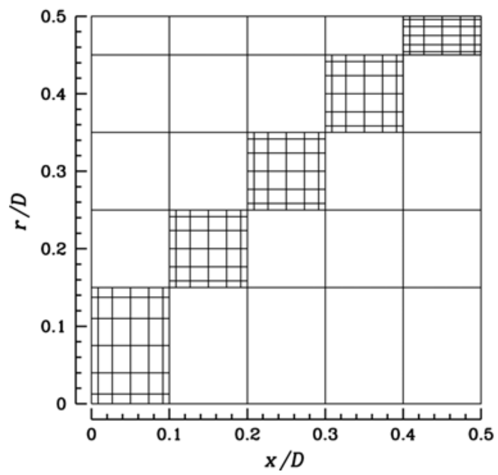


Figure 2: An example of a spectral element mesh in the meridional semi-plane for the cylindrical-coordinate formulation used here. Internal node points are drawn in selected elements for a tensor-product polynomial shape function order N=6. The axial domain length corresponds to an axial wavenumber $\alpha = 2\pi/0.5 = 12.57$.

RESULTS

Comparison to data of Yang & Yih (1977)

We first examine stability to axisymmetric perturbations (k = 0), as dealt with previously by Yang & Yih (1977). Yang & Yih found that axially invariant modes (α = 0) were the least stable, and provided results for a range of Reynolds numbers, dimensionless frequencies, and axial wavenumbers. We have re-interpreted their dimensionless groups as Womersley number and reduced velocity. A comparison between their results and ours for α = 0, k = 0 is shown in Figure 3. In all cases, the flow is stable ($\mu < 1$), but only marginally so at large *Wo*.

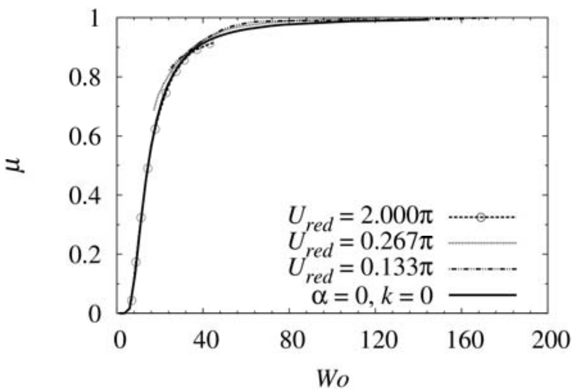


Figure 3. Floquet multiplier data derived from Yang & Yih (1977) figure 1, for the dominant axially invariant axisymmetric modes. Compilation of comparable results obtained from present computations shown as a solid line (α = 0, k = 0).

Our data compare well with those from Yang & Yih, but also consideration of this figure brings out a significant point that has previously been unremarked. Through not adopting Womersley number as a dimensionless group with which to represent their data, Yang & Yih apparently did not notice that for axially invariant modes, it becomes the only active parameter and collapses three sets of their data around a single curve (the slight discrepancies seen are attributable noise in our digitization of their figures). We have confirmed this by running analyses at a wide range of the two dimensionless parameters, and find that all our leading multiplier data for α = 0, k = 0, fall on a single curve when plotted against Womersley number.

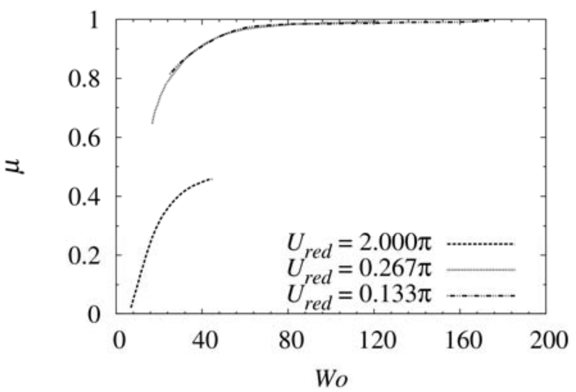
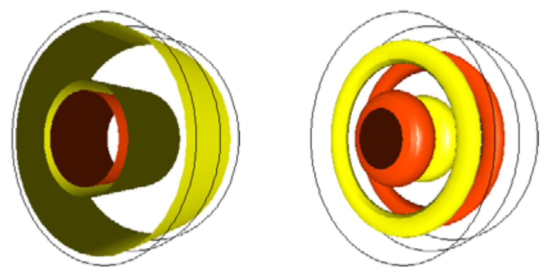


Figure 4. Yang & Yih (1977) dominant Floquet multiplier data for axisymmetric modes (k = 0) with α = 4. Note that here (as opposed to Figure 3) there is no collapse with Womersley number.

We note that this collapse is not seen in their data for non-axially invariant modes, i.e. $\alpha > 0$. This may be observed in Figure 4, where no collapse of data with Womersley number is found. Our results, to be discussed below, also show that for $\alpha > 0$, there are (as expected *a priori*) again two dimensionless groups. We can explain this effect using the following physical reasoning. Axially invariant modes do not see the influence of the kinematic parameter U_{red}, which as we previously noted describes how far the oscillatory flow moves back and forth along the pipe. This leaves only *Wo* as a parameter. However, for $\alpha > 0$, the modes also have a finite axial dimension and this allows an interaction so that stability is characterized both by *Wo* and U_{red}.

For purposes of comparison, we show in Figure 5 illustrative examples of both axially invariant and axially variant (but axisymmetric) Floquet mode shapes These are drawn for one phase point in the base flow cycle. Both mode shapes are obtained at *Wo* = 97, U_{red} = 2.5. We note that for k = 0, α = 0, continuity requires Floquet modes in which only axial velocity components can be non-zero.



(a) k = 0, α = 0. (b) k = 0, α = 16.

Figure 5 Comparison of axisymmetric (k = 0) Floquet mode shapes; (a) axially invariant, α = 0; (b) with axial wavenumber α = 16. Visualized as isosurfaces of positive/negative axial velocity component.

Stability of axially invariant modes (α = 0)

Yang & Yih (1977) dealt only with the stability of axisymmetric modes, k = 0. Since there is no known equivalent to Squire’s theorem (Squire 1933) in cylindrical coordinates, there is no reason to expect *a priori* that axisymmetric modes are the least stable. We commence examination of results for general modes by studying the effect of azimuthal wavenumber k on axially invariant modes, α = 0; as explained above. The only remaining parameter in these cases is the Womersley number.

Figure 6 shows that axisymmetric modes are in fact the least stable, since Floquet multipliers reduce monotonically in magnitude with increasing azimuthal wavenumber at all values of *Wo*. This shows that all axially invariant modes are stable.

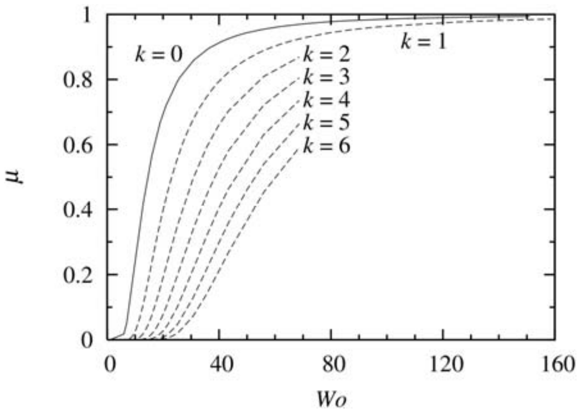
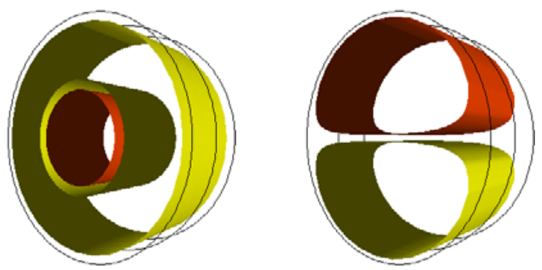


Figure 6 Dominant Floquet multipliers for axially invariant modes, $\alpha = 0$, both axisymmetric ($k = 0$) and non-axisymmetric ($k > 0$) as functions of Womersley number.

Figure 7 illustrates the distinction between axisymmetric and non-axisymmetric Floquet modes, but where both are axially invariant, $\alpha = 0$. Again we note that since these modes are axially invariant, the only non-zero velocity component must be in the axial direction.



(a) $k = 0$, $\alpha = 0$. (b) $k = 1$, $\alpha = 0$.

Figure 7 Comparison of (a) axisymmetric ($k = 0$) and (b) non-axisymmetric ($k = 1$) Floquet mode shapes, both axially invariant, $\alpha = 0$. Visualized as isosurfaces of positive/negative axial velocity component. Computed at $Wo = 97$.

To demonstrate that the dominant axially invariant, axisymmetric mode is significantly less stable than the subdominant modes, we show in Figure 8 the Floquet multipliers of the three leading modes for $\alpha = 0$, $k = 0$.

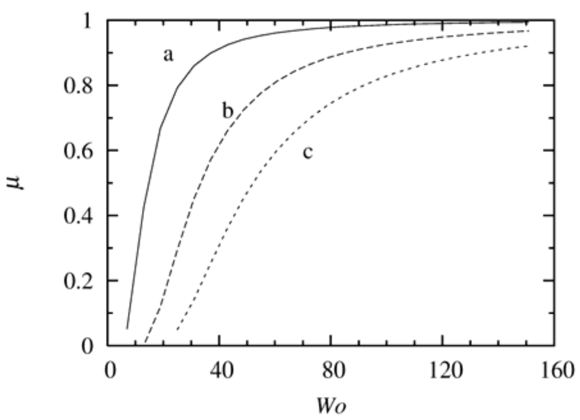


Figure 8 Floquet multipliers for the leading three (*a*, *b*, *c*) axially invariant and axisymmetric modes ($\alpha = 0$, $k = 0$). Curve *a* is the same as the curve labelled $k = 0$ in Figure 6.

Stability of axially variant modes ($\alpha > 0$)

Next we turn to examine the question of whether modes with axial structure are more, or less, stable than those without. Recall that in these cases, we expect both Womersley number and reduced velocity as control parameters. We note that Yang and Yih's results suggest that for axisymmetric modes, axially invariant cases ($\alpha = 0$) are the least stable for a fixed value of reduced velocity. As shown in Figure 9, our results also support this conclusion. At fixed Womersley number, Floquet multipliers decrease monotonically as axial wavenumber is increased (axial wavelength is decreased). A comparison of typical mode shapes for axisymmetric axially invariant and axially variant modes was shown in Figure 5.

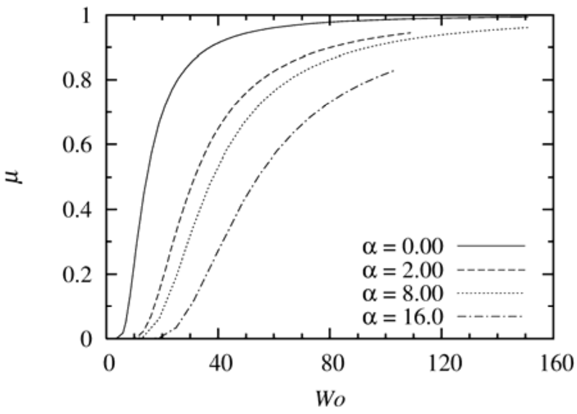


Figure 9 Dominant Floquet multipliers for axisymmetric modes ($k = 0$) at $U_{red} = 2.5$, showing the effect of axial wavenumber.

In order to demonstrate that there are in fact two control parameters when mode shapes are axially variant, we show in Figure 10 the leading Floquet multipliers for axisymmetric modes ($k = 0$) at a fixed axial wavenumber $\alpha = 16$. For axially invariant modes, we re-iterate that all results would collapse to a single function of Womersley number. Here, two distinct curves are obtained, one for each reduced velocity.

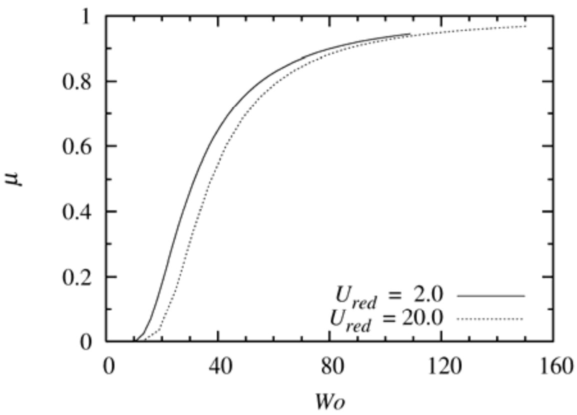


Figure 10 Dominant Floquet multipliers for axially variant but axisymmetric modes, with two values of reduced velocity. These values are computed at fixed $\alpha = 16$.

CONCLUSION

The present study extends the work of Yang & Yih (1977), who examined the linear stability of axisymmetric disturbances to oscillatory pipe flows. Our new contribution here is the extension to non-axisymmetric disturbances, however these are generally more stable than equivalent axisymmetric cases. In agreement with Yang & Yih we have found that the least stable disturbances are provided by axially invariant modes.

A significant new result, not recognised in past work, is that the stability of these axially invariant modes is dependent only upon a single control parameter, the Womersley number, which is also the only parameter needed to describe the radial velocity profiles of the base flows. When modes with axial variation are considered, a second control parameter comes into effect. We have used the reduced velocity as this second control parameter – this can be interpreted as a measure of the number of pipe diameters that the oscillatory bulk flow travels in one oscillation period. Modes without axial variation (thus, without an axial length scale) do not respond to changes in reduced velocity. Base flows with larger reduced velocities are more stable than those with smaller reduced velocities.

Our study suggests that all single-harmonic oscillatory pipe flows are linearly stable, but approach instability asymptotically with increasing Womersley number. This is analogous to the behaviour of steady pipe flows, which are observed to be linearly stable at all finite Reynolds numbers, but less stable at high Reynolds number.

Finally, we note (apparently for the first time) that since all periodic laminar pulsatile flows can be described as the superposition of steady flow and oscillations at different temporal harmonics, it suffices to study separately the linear stability of each of the temporal harmonics. Since steady pipe flow is well-known to be linearly stable at all finite Reynolds numbers, and our results show that single-harmonic oscillatory pipe flows are stable to both axisymmetric and non-axisymmetric disturbances at all finite values of the two control parameters, we have the general result that all time-periodic laminar pulsatile flows in circular pipes are linearly stable.

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