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STRESS IN SANDPILES

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ABSTRACT

We investigate the stress distribution at the base of a sand pile using both analytic calculations and a three dimensional discrete element code. We consider the controversial subject of the depression in the normal stress under the highest part of a sand pile. We find that piles composed of particles with the same size do not produce a stress depression, while piles composed of layers of particles of different sizes do show a stress depression which is similar to experimental observations.

NOMENCLATURE

f_{zj} the normalized stress
g acceleration due to gravity
H height of the sandpile
m_p particle mass
q number of horizontal layers in a tetrahedral pile
r the radial distance from the centre-line of the pile
S_q^b number of base particles for a "q" tetrahedral pile
S_q^t number of particles in a "q" tetrahedral pile

Δ deflection at the base of the sandpile
ϕ set of non-dimensional dependent variables
ρ bulk density of the granular material
σ_{zj} component of stress

INTRODUCTION

The study of stress distributions in granular materials has been a subject of research for at least seventy years. Most of the research activity has been conducted in the engineering community, where investigators have been interested in the stability of earth embankments, rock piles and dams (Hummel and Finnan, 1920; Brahtz, 1936; Trollope, 1956, 1968; Burman ,1971; Penman, 1986).

In the last decade, however, the unusual behaviour of granular materials has inspired an upsurge of interest from other scientific communities, such as applied mathematics and physics (Hong, 1993; Liffman *et al.* 1992, 1994; Luding, 1997; Oron and Herrmann, 1998, Huntley, 1998).

Much of the most recent research has been inspired by the the experiments of Smid and Novosad (1981) and Jotaki and Moriyama (1979), who created conical piles by pouring granular material onto a measuring table equipped with pressure cells and thereby observed, in many cases, a pressure minimum under the highest part of the pile.

These new research communities have usually been unaware of past research done by the engineering community (Savage, 1997). For example, Trollope and coworkers have shown, both by experimental and theoretical analysis, that a stress dip can appear in a pile of granular material when the base of pile is allowed to sag (Trollope, 1968).The deflection of the base required to give an appreciable dip is quite small, e.g., if Δ is the deflection of the base and H is the height of the pile, then experiments on a granular wedge show that $\Delta/H \approx 2\%$ gives an appreciable dip in the normal stress Trollope (1956). For this, and other reasons, the engineering community tend to regard deflection or sag at the base of pile as the major cause of the stress dip.

It is possible, however, that the stress dip may appear for reasons other than just a sagging base. There have been a number of studies where a stress dip appears under a conical pile of material, but a sagging base may not have been responsible for this effect. For example, in the experiments undertaken by Brockbank *et al.* (1997), the contact zone between a layer of ball bearings and a transparent rubber sheet was used to determine the pressure distribution under piles of different granular materials. This is a somewhat different experiment from past work, because the measured Δ/H values refer to the individual ball bearings and thus to the local values of the stress at the base of the pile. In prior work, Δ/H referred to the sag at the base, which is more like an imposed boundary condition at the base of the pile.

For all the experiments undertaken by Brockbank *et al.* (1997), Δ/H was approximately 10^{-4}, yet some material - such as sand, "small" glass beads and a mixture of small and large glass beads - showed the stress dip, while other materials - lead shot, large glass beads with different friction coefficients, and flour - did not. In our view, this indicates that different physical mechanisms influence stress within a pile and that another physical mechanism, besides a sagging base, is responsible for the observed stress depression.

In this study, we show, numerically, that the stress dip can arise from particle segregation. This possibility was first suggested by Liffman *et al.* (1994), who were able to obtain a dip in the normal stress by placing larger particles above smaller particles in a numerical 2D sandpile. In reality, however, such a segregation pattern is not found in conical sandpiles. If one pours polydisperse granular material (*i.e.*, material containing particles of different sizes) from a point source on to a flat base to create a conical sandpile, larger particles do not go to the top of the pile. Instead, the sandpile may obtain a layered internal structure, (see Figure 1) , where one layer contains the smaller particles while the adjacent layer contains the larger particles. This form of segregation is often called ``the Christmas Tree Effect".

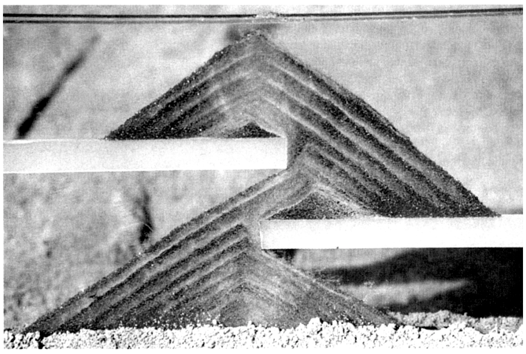

Figure 1. The Christmas Tree Effect. Separated layers of granular material formed by pouring a mixture of two differently sized particles between two glass plates. The smaller particles form the light layers the larger particles forming the dark layers. Picture courtesy of H. A. Makse

The reasons why this segregation occurs are slowly becoming understood. For example, it is known that the size and frictional properties of the particles can play a part in aiding or inhibiting this segregation (Makse *et al.* 1997, Makse 1999). Kinematic processes also appear to play a role, since a "slow" pour (Baxter *et al.* 1998), may create a "Christmas Tree" pile, while a "fast" pour will create a pile where the small particles are located in the central core of the pile, and the larger particles are found in the rest of the pile.

Recently, Wittmer *et al.* (1996,1997) have questioned the link between size segregation and the observed dip in the normal stress under a pile. As has been noted in the wider scientific press Watson (1996), Wittmer *et al.* have developed their own stress model to explain the stress depression phenomenon. One of the fundamental assumptions in their model is that "there is no characteristic length-scale intrinsic to the granular medium of which a pile is made", thus - it is claimed -size segregation could not cause the stress dip, because such segregation would introduce a length scale by setting up gradients in the material properties of the pile. The experimental data cited in Wittmer *et al.*, to justify their no-scale hypothesis, shows that stress in (some) sandpiles *scales with the height of the pile*. This result also appears in the experimental work of Brockbank *et al*. (1997). This implies that stress in some sandpiles satisfies the equation

$$\sigma_{zj} = gH\rho f_{zj}(\frac{r}{H},\phi). \qquad (1)$$

It is not surprising that the height of the pile should appear in a "no-scale" system. The length scales in such systems tend to disappear from the solutions only when the system has an infinite size. A sandpile, however, is a finite system, where the relevant length scale is the width of the base or, equivalently, the height of the pile.

From Figure 1, it would appear that size segregation in sandpiles can also scale with the height of the pile. For example the number of layers in the pile is proportional to the height of the pile, or the height of the fine particle core is equal to the height of the pile. Given this behaviour, we believe that the rejection of our size segregation hypothesis by Wittmer *et al.* may be premature.

To investigate the stress distributions in conical piles with particle size segregation/stratification, we have constructed a 3D discrete element code, a detailed description of which is given in Liffman *et al.* (2000). Using this code we can examine the internal force structure of conical piles of particulate material where the piles, typically, contain around 10,000 particles. Although one can calculate the stress tensors within the pile (Goddard, 1986; Bathurst, 1988; Luding, 1997), for reasons of brevity, we will only consider the force structure between the particles. We hope to examine the stress patterns within a pile in a later, detailed study.

To ensure that our code is working correctly, we have developed an analytic model of the force structure in a 3D tetrahedral pile. A detailed derivation of this model is given in Liffman *et al.* (2000). In this paper, we outline this analytic model and then compare simulation with theory. In the final sections of this paper, we construct a conical sandpile and show how size segregation can produce a depression in the normal stress at the base of the sandpile.

AN ANALYTIC MODEL

We consider a tetrahedral pile of identical spheres, where the pile has a face-centered cubic lattice packing. We describe these tetrahedral piles by a q number, where q represents the number of horizontal layers in the pile. A q = 18 pile is shown in Figure 2

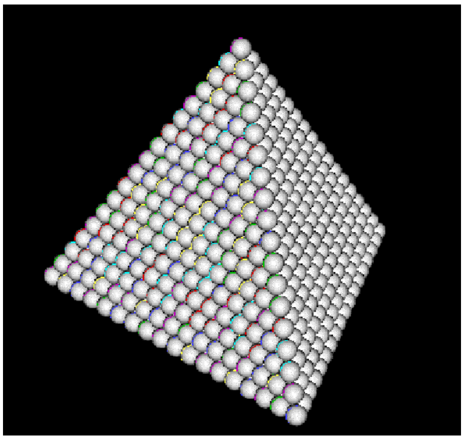

Figure 2. A q = 18 tetrahedral pile consisting of 1140 particles

The total number of particles in a tetrahedral pile of q layers, S_q^t, is

$S_q^t = q(q+1)(2q+4)/12$. (2)

Similarly, the number of particles at the base of the pile, S_q^b, is given by

$S_q^b = q(q+1)/2$. (3)

As was first derived by Huntley (1998), and is also shown in Liffman *et al.* (2000), the normal force (or pressure) at the base of such a pile is a constant for all particles at the base of the pile. So, the weight force per particle at the base of the pile, W_q^b, is simply

$$W_q^b = m_p g(q+2)/3 \;. \qquad (4)$$

A more rigorous derivation of Eq. (3) is given in Liffman *et al.* (2000), where we also derive the expected horizontal shear forces at the base of the pile.

SIMULATION VERSUS ANALYTIC MODEL

To investigate the stress distribution in granular material, we built a three dimensional, discrete element model (DEM). In this code, we used slightly deformable spheres as a model for real particles, where we computed the forces acting on the particles and integrated Newton's equations of motion to determine the subsequent motion of each of the particles. There were damped, centre to centre forces between the particles, but no interparticle, tangential frictional forces. Static and slip friction only arose between the particles and the ground. The particles had a mass density of 3 g cm^{-3} and were subject to the standard gravitational acceleration, g. A detailed description of the code is given in Liffman *et al.* (2000).

To demonstrate the veracity of our code, we can compare the analytic solutions of the tetrahedral pile (Liffman *et al.* 2000) with the results computed from our code for a tetrahedral pile of spheres. In particular we examine the forces in the q = 18 tetrahedral pile shown in Figure 2

From Eq. (4), each particle at the base of the q = 18 pile should be subject to a normal force of 20/3 $m_p g$. In Figure 3, we show the numerical versus theoretical results. Of the two surfaces shown in Figure 3 the plane represents the analytic result, while the curved surface represents the numerical simulation.

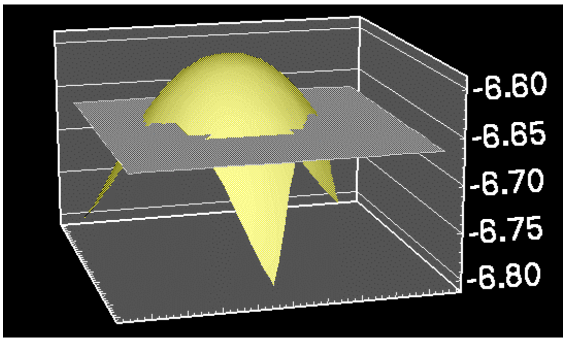

Figure 3. Simulation versus theory for the normal force (in units of particle weight) at the base of the pile. The flat plane shows the theoretically expected result, the curved surface the simulation result. The maximum difference between theory and simulation is < 1%.

At first sight, the divergence between the two results would suggest considerable error in the numerical simulation. In fact, there is excellent agreement between theory and simulation, where the maximum deviation between the computer model and analytic theory is < 1%. We have magnified the difference for visualization purposes. The observed difference is due to our "analytic particles" being infinitely hard, while our DEM particles are deformable. We can easily reduce the difference between theory and simulation by making our computer particles harder and less deformable, but an error of < 1% is more than adequate for our purposes.

We have also derived analytic solutions for the forces within a tetrahedral pile and shear forces at the base of such a pile. For such cases, our comparisons between theory and computer simulation again show agreement within our set tolerance level of 1% (Liffman *et al.* 2000).

THE PRESSURE DIP

To determine the force structure in and under a conical pile of spherical particles, we created a pile that contained 10,331 particles and had an approximate angle of repose of 30^o (Figure 4). The particles had a mass density of 3 g cm^{-3} and were subject to standard gravity.

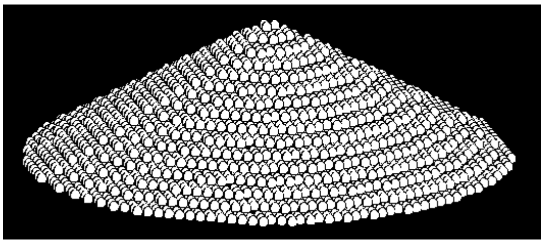

Figure 4. A conical pile consisting of 10,331 particles.

Using this basic system, we constructed two sandpiles. In the first sandpile all 10,331 particles had the same diameter of 1 mm. We called this pile the "homogeneous" sandpile. In the second pile, using Figure 1 as a guide ,we placed a vein of slightly larger particles inside the pile. We denoted this as the "inhomogeneous" pile. A cross-section of the inhomogeneous pile is shown in Figure 5. The vein of large-particles was a three dimensional structure with a conical shape and a 30^o angle of repose. To make this layer, we simply selected the appropriate particles within the homogeneous pile and increased their size by the same amount. The diameters of the larger particles were typically set at ≤ 1.005 mm.

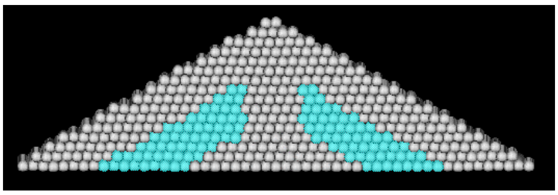

Figure 5. Cross section through the inhomogeneous conical pile showing a layer of larger particles

The presence of the large-particle layer produced a dramatic change in the force structure of the pile. This is shown in the Figures 6 and 7, where we display a 3D rendering of the normal force at the base of the homogeneous and inhomogeneous piles.

The homogeneous sandpile had a smooth normal-force surface with the (naively) expected structure of the maximum normal-force occurring under the highest part of the pile (Figure 6).

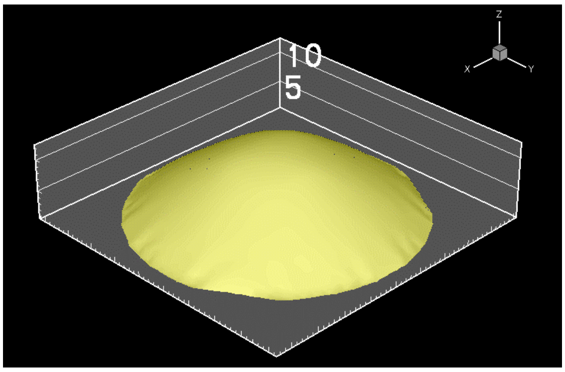

Figure 6. 3D rendering of the smooth normal force structure (in units of particle weight) at the base of the homogeneous sandpile.

The presence of slightly larger particles (in the inhomogeneous pile) produces a sizeable force dip under the highest part of the sandpile (Figure 7).

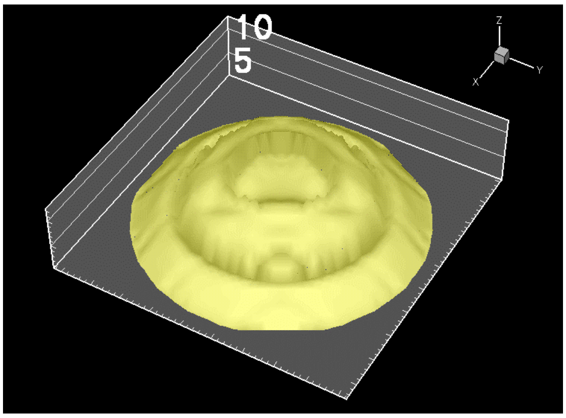

Figure 7. 3D rendering of the dipped normal force structure at the base of the inhomogeneous sandpile

A comparison between the two surfaces is shown in Figure 8, where we have taken a cross section of the normal force surface along a base line of the pile that passes through the central axis of the pile. In this figure (and Figure 9), the normal forces and distances are given in terms of the particle weight and particle diameter (1 mm), respectively. The unbroken line shows the force structure for the homogeneous pile, while the broken line shows the significant force dip induced by the presence of a vein of large particles in the inhomogeneous pile.

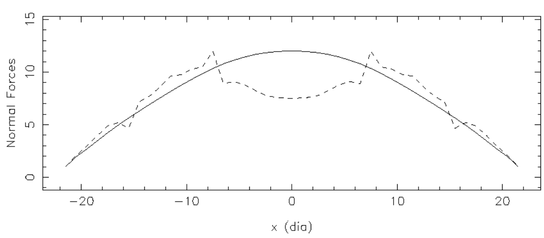

Figure 8. 2D cross section of the normal force structures in the homogeneous (smooth line) and inhomogeneous (broken line) piles. The inhomogeneous pile has a significant depression in the normal force relative to the homogeneous pile. Force units are in particle weight.

By changing the size and position of the larger particles, we were able to change the shape of the force dip. For example, the depth of the force dip was approximately proportional to the diameter of the larger particles, while the width of the force depression was dependent on the length of the vein of large particles.

For the data shown in Figure 8, we set the size of the larger particles to 1.005 mm, and used the vein shape shown in Figure 5. In this case, the maximum force occurred at 1/3 of the pile's radius from the centre of the pile, while the minimum of the normal force dip was approximately 40% of the maximum value of normal force. These values are similar to what is observed experimentally (Smid and Novosad, 1981; Jotaki and Moriyama, 1979; Brockbank *et al.* 1997, Vanel *et al.* 1999), but the shape of the force dip in Figure 8 is different from what is observed for real sandpiles. In real sandpiles, the force dip has a relatively sharp minimum on or near the central axis of the pile. In our inhomogeneous pile, the force dip occurs over a large central region. We are currently trying different size segregation patterns in an attempt to obtain a more realistic result.

These preliminary results suggest that the inclusion of larger particles within the pile increases the horizontal shear stress within the pile. An example of this effect is shown in Figure 9, where we display the horizontal shear stress along the same base line as in Figure 8.

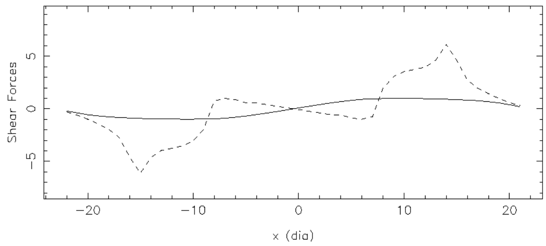

Figure 9. Horizontal shear stress at the base of the homogeneous (unbroken line) and inhomogeneous (broken line) piles. Force units are in particle weight

In Figure 9, the peak horizontal shear stress at the base of the inhomogeneous pile is approximately six times larger than the shear stress for the homogeneous pile. This result is similar to that obtained in Trollope's "clastic" model (Trollope, 1968). Trollope showed, analytically and numerically, that the inclusion of horiztontal shear stress within the pile produced a significant dip in the normal stress at the base of the pile. Trollope applied his model to the traditional engineering case of a sandpile with a sagging base, but his analysis can equally be applied to our case, where the size segregation of larger particles within the pile can produce increased horizontal shear stress within the pile.

CONCLUSIONS

The dip in the normal stress under the highest part a sandpile is a relatively well known phenomenon that has been investigated by engineers since the 1920's. This "arching" phenomenon often occurs when the base of the pile is allowed to sag. As a result of these studies, engineers tend to assume that a dip in the normal stress, under a sand pile, is indicative of a sagging base. There are experimental results, however, which show the existence of a stress dip even when the base of the pile is relatively firm (*e.g.,* Brockbank *et al.* 1997, Vanel *et al.* 1999). We suggest that, in such cases, the force dip may be due to size segregation within the pile.

When material is poured from a point source to form a conical pile, size segregation tends to produce a sand pile where the small particles are in the central core of the pile and the larger particles are in the rest of the pile. A layered "Christmas Tree" pattern can also arise within the pile where the large and small particles self-organise into different, inclined layers (Figure 1). To investigate the possible forces that may be produced by such size segregation, we have constructed numerical sandpiles with a 3D Discrete Element Model.

Two such piles were created, one with and one without an internal layer of larger particles. There was a marked difference between the normal force structure between the two piles. The inhomogeneous pile - with the layer of larger particles - had a dip in the normal stress which was similar to that observed in experiments. The homogeneous pile - with uniformly-sized particles - showed no such dip in the normal stress. Indeed, this latter pile had a maximum normal force under the highest part of the pile.

Our preliminary examination of the forces in these piles suggest that size segregation increases the horizontal shear stress within the pile, which in turn increases the normal stress at the base of the pile.

This preliminary conclusion is similar to that obtained by Trollope (Trollope, 1968), whose "clastic model" for the stress distribution in granular solids readily produces a stress dip, where the magnitude of the stress dip is somewhat proportional to the magnitude of the horizontal shear stresses with the pile.

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